

# Exact Seiberg-Witten Map and Induced Gravity from Noncommutativity

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## ABSTRACT

We find a closed form for Seiberg-Witten (SW) map between ordinary and noncommutative (NC) Dirac-Born-Infeld actions. We show that NC Maxwell action after the exact SW map can be regarded as ordinary Maxwell action coupling to a metric deformed by gauge fields. We also show that reversed procedure by inverse SW map leads to a similar interpretation in terms of induced NC geometry. This implies that noncommutativity in field theory can be interpreted as field dependent fluctuations of spacetime geometry, which genuinely realizes an interesting idea recently observed by Rivelles.

Keywords: Noncommutative field theory; Exact Seiberg-Witten map; Dirac-Born-Infeld action

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# 1 Dirac-Born-Infeld Action

We revisit here the equivalence between noncommutative (NC) and ordinary gauge theories discussed in [1]. We leave the geometry of spacetime background fixed and concentrate, instead, on the dynamics of open string sectors of the theory. To be specific, we consider open strings attached on  $Dp$ -branes in flat spacetime, with metric  $g_{\mu\nu}$ , in the presence of a constant Neveu-Schwarz  $B$ -field.<sup>1</sup> We define a parameter describing the size of a string as

$$\kappa \equiv 2\pi\alpha', \quad (1.1)$$

which is a useful expansion parameter in low energy effective action of D-branes. The worldsheet action is

$$\begin{aligned} S &= \frac{1}{2\kappa} \int_{\Sigma} d^2\sigma (g_{\mu\nu} \partial_a x^\mu \partial^a x^\nu - i\kappa B_{\mu\nu} \varepsilon^{ab} \partial_a x^\mu \partial_b x^\nu) - i \int_{\partial\Sigma} d\tau A_\mu(x) \partial_\tau x^\mu \\ &= \frac{1}{2\kappa} \int_{\Sigma} d^2\sigma g_{\mu\nu} \partial_a x^\mu \partial^a x^\nu + i \int_{\partial\Sigma} d\tau \left( \frac{1}{2} B_{\mu\nu} x^\nu - A_\mu(x) \right) \partial_\tau x^\mu, \end{aligned} \quad (1.2)$$

where string worldsheet  $\Sigma$  is the upper half plane parameterized by  $-\infty \leq \tau \leq \infty$  and  $0 \leq \sigma \leq \pi$  and  $\partial\Sigma$  is its boundary. The propagator evaluated at boundary points [1] is

$$\langle x^\mu(\tau) x^\nu(\tau') \rangle = -\frac{\kappa}{2\pi} \left( \frac{1}{G} \right)^{\mu\nu} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{\mu\nu} \epsilon(\tau - \tau') \quad (1.3)$$

where  $\epsilon(\tau)$  is the step function. Here

$$\left( \frac{1}{G} \right)^{\mu\nu} = \left( \frac{1}{g + \kappa B} g \frac{1}{g - \kappa B} \right)^{\mu\nu}, \quad (1.4)$$

$$G_{\mu\nu} = g_{\mu\nu} - \kappa^2 (B g^{-1} B)_{\mu\nu}, \quad (1.5)$$

$$\theta^{\mu\nu} = -\kappa^2 \left( \frac{1}{g + \kappa B} B \frac{1}{g - \kappa B} \right)^{\mu\nu}. \quad (1.6)$$

From Eqs. (1.4) and (1.6), we have the following relation

$$\frac{1}{G} + \frac{\theta}{\kappa} = \frac{1}{g + \kappa B}. \quad (1.7)$$

The object  $G_{\mu\nu}$  has a simple interpretation as the effective metric seen by the open strings while  $g_{\mu\nu}$  is the closed string metric. Furthermore the coefficient  $\theta^{\mu\nu}$  has a simple interpretation as

$$[x^\mu(\tau), x^\nu(\tau)] = i\theta^{\mu\nu}. \quad (1.8)$$

That is,  $x^\mu$  are coordinates on a NC space with noncommutativity parameter  $\theta$  [2, 3, 4, 5, 6].

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<sup>1</sup>Here we will take  $g_{\mu\nu}$  with either Lorentz or Euclidean signature since the signature is inconsequential in our discussions. Also we take  $B_{\mu\nu}$  with rank  $r = p + 1$ ,  $\mu, \nu = 0, 1, \dots, p$ , for simplicity, although our discussion well applies to the case  $r < p + 1$ .

For a slowly varying approximation of neglecting derivative terms, i.e.,  $\sqrt{\kappa}|\frac{\partial F}{F}| \ll 1$ , the open string effective action on a D-brane was shown to be given by the Dirac-Born-Infeld (DBI) action [7, 8]. Seiberg and Witten, however, showed [1] that an explicit form of the effective action depends on the regularization scheme of two dimensional field theory defined by the worldsheet action (1.2), which is related to field redefinitions in spacetime.

A sigma model path integral with Pauli-Villars regularization preserves the ordinary gauge symmetry of open string gauge fields. With such a regularization, the effective action of a D-brane can depend on  $B$  and  $F = dA$  only in the combination  $F + B$ , since there is a symmetry  $A \rightarrow A + \Lambda$ ,  $B \rightarrow B - d\Lambda$ , for any one-form  $\Lambda$ . In this case, the spacetime low energy effective action on a single  $Dp$ -brane is given by the DBI action

$$S(g_s, g, A, B) = \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{-\det(g + \kappa(F + B))}, \quad (1.9)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (1.10)$$

Note that the effective action is expressed in terms of closed string variables  $g_{\mu\nu}$ ,  $B_{\mu\nu}$  and  $g_s$ .

With a point-splitting regularization [1], the spacetime effective action is expressed in terms of NC gauge fields and has the NC gauge symmetry on the NC spacetime defined by Eq. (1.8). In this description, the analog of Eq. (1.9) is

$$\hat{S}(G_s, G, \hat{A}, \theta) = \frac{2\pi}{G_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{-\det(G + \kappa\hat{F})}. \quad (1.11)$$

The action depends on the open string variables  $G_{\mu\nu}$ ,  $\theta_{\mu\nu}$  and  $G_s$ , where the  $\theta$ -dependence is entirely in the  $\star$  product in the field strength  $\hat{F}$ :

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu \star \hat{A}_\nu + i\hat{A}_\nu \star \hat{A}_\mu. \quad (1.12)$$

The DBI action (1.11) is definitely invariant under

$$\hat{\delta}_{\hat{\lambda}} \hat{A}_\mu = \widehat{D}_\mu \star \hat{\lambda} = \partial_\mu \hat{\lambda} - i\hat{A}_\mu \star \hat{\lambda} + i\hat{\lambda} \star \hat{A}_\mu. \quad (1.13)$$

The ambiguity related to the choice of regularization scheme is a well-known field redefinition ambiguity present in the effective action reconstructed from S-matrix. Thus the two descriptions with different regularizations should be related by a spacetime field redefinition. Indeed, Seiberg and Witten found a transformation from ordinary to NC gauge fields in a way that preserves the gauge equivalence relation between ordinary and NC gauge symmetries [1]. The Seiberg-Witten (SW) map relating the gauge potentials and field tensors to the first order in  $\theta$  is given by

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}) + \mathcal{O}(\theta^2), \quad (1.14)$$

$$\hat{F}_{\mu\nu} = F_{\mu\nu} + \theta^{\alpha\beta} (F_{\mu\alpha} F_{\nu\beta} - A_\alpha \partial_\beta F_{\mu\nu}) + \mathcal{O}(\theta^2). \quad (1.15)$$

Since the commutative and NC descriptions arise from the same open string theory depending on different regularizations and the physics should not depend on the regularization scheme, Seiberg and Witten [1] argued that<sup>2</sup>

$$\hat{S}(G_s, G, \hat{A}, \theta) = S(g_s, g, A, B) + \mathcal{O}(\sqrt{\kappa}\partial F). \quad (1.16)$$

The equivalence (1.16) may also be understood by different path integral prescriptions for open strings ending on a D-brane [9, 10]. If  $B$  field is constant, the term involving the  $B$  field in the action (1.2) can be treated as a part of kinetic term or as a part of boundary interaction, since it is quadratic in string variables. In the former case we get the NC DBI action (1.11) and in the latter case the ordinary one (1.9). Since the two are obtained by evaluating the same Polyakov string path integral, it establishes that the NC DBI action is equivalent to the ordinary one.

First of all, the equivalence (1.16) determines the open string coupling constant  $G_s$  by demanding that for  $F = \hat{F} = 0$  the constant terms in the actions using the two set of variables are the same:

$$G_s = g_s \sqrt{\frac{\det G}{\det(g + \kappa B)}}. \quad (1.17)$$

In the comparison (1.16), the action  $\hat{S}(G_s, G, \hat{A}, \theta)$  is expressed in terms of open string parameters while  $S(g_s, g, A, B)$  is in terms of closed string parameters. For an explicit comparison, we will use the same string variables for two different descriptions. First, we reexpress Eq. (1.9) in terms of open string variables using the conversion relations, Eqs. (1.7) and (1.17), between open and closed string parameters

$$S(G_s, G, A, \theta) = \frac{2\pi}{G_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{-\det(G + F\theta G + \kappa F)}. \quad (1.18)$$

In what follows, we will often use the matrix notation

$$AB = A_{\mu\alpha}B^{\alpha\mu}, \quad (AB)_{\mu\nu} = A_{\mu\alpha}B^{\alpha}_{\nu}, \quad \text{etc.} \quad (1.19)$$

Later we will also consider the equivalence in terms of closed string variables.

As was explained in [1], there is a general description with an arbitrary  $\theta$  associated with a suitable regularization that interpolates between Pauli-Villars and point-splitting. This freedom is basically coming from the fact that the gauge invariant combination of  $B$  and  $F$  in open string theory is  $\mathcal{F} = B + F$ . Thus there is a symmetry of shift in  $B$  keeping fixed  $B + F$ . Given such a symmetry, we may split the  $B$  field into two parts and put one in kinetic part and the rest in boundary interaction part. By taking the background to be  $B$  or  $B'$ , we should

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<sup>2</sup>As already pointed out in [1], the comparison cannot be made for constant  $F$  since terms in Eq. (1.15) such as the form  $A\partial F$  contribute in the analysis. It is necessary to integrate by parts in comparing the DBI actions, and one cannot naively treat  $F$  as a constant.

get a NC description with appropriate  $\theta$  or  $\theta'$ , and different  $\hat{F}$ 's. Hence we can write down a differential equation that describes how  $\hat{A}(\theta)$  and  $\hat{F}(\theta)$  should change when  $\theta$  is varied, to describe equivalent physics [1]:

$$\delta\hat{A}_\mu(\theta) = -\frac{1}{4}\delta\theta^{\alpha\beta}(\hat{A}_\alpha \star (\partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu}) + (\partial_\beta \hat{A}_\mu + \hat{F}_{\beta\mu}) \star \hat{A}_\alpha), \quad (1.20)$$

$$\begin{aligned} \delta\hat{F}_{\mu\nu}(\theta) = & \frac{1}{4}\delta\theta^{\alpha\beta}(2\hat{F}_{\mu\alpha} \star \hat{F}_{\nu\beta} + 2\hat{F}_{\nu\beta} \star \hat{F}_{\mu\alpha} - \hat{A}_\alpha \star (\widehat{D}_\beta \hat{F}_{\mu\nu} + \partial_\beta \hat{F}_{\mu\nu}) \\ & - (\widehat{D}_\beta \hat{F}_{\mu\nu} + \partial_\beta \hat{F}_{\mu\nu}) \star \hat{A}_\alpha). \end{aligned} \quad (1.21)$$

Incidentally Eq. (1.14) and Eq. (1.15) are a solution of the differential equations (1.20) and (1.21) to first order in  $\theta$ , respectively. An exact solution of the differential equation (1.21) in the Abelian case was given in [11, 12, 13, 14]. Especially, for the case of rank one gauge field with constant  $\hat{F}$ , the equation (1.21) can be easily solved to be

$$\hat{F} = \frac{1}{1 + F\theta} F. \quad (1.22)$$

The freedom in the description just explained above is parameterized by a two-form  $\Phi$  from the point of view of NC geometry on the D-brane worldvolume. In this case the change of variables found by Seiberg and Witten [1] is given by

$$\frac{1}{\mathcal{G} + \kappa\Phi} + \frac{\vartheta}{\kappa} = \frac{1}{g + \kappa B}, \quad (1.23)$$

$$\mathcal{G}_s = g_s \sqrt{\frac{\det(\mathcal{G} + \kappa\Phi)}{\det(g + \kappa B)}}. \quad (1.24)$$

The effective action in these variables are modified to

$$\hat{S}_\Phi(\mathcal{G}_s, \mathcal{G}, \hat{A}, \vartheta) = \frac{2\pi}{\mathcal{G}_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{-\det(\mathcal{G} + \kappa(\hat{\mathcal{F}} + \Phi))}. \quad (1.25)$$

For every background characterized by  $B, g_{\mu\nu}$  and  $g_s$ , we thus have a continuum of descriptions labelled by a choice of  $\Phi$ . Indeed, for  $\Phi = B$  where  $\mathcal{G} = g$ ,  $\mathcal{G}_s = g_s$  and  $\vartheta = 0$ ,  $\hat{S}_\Phi$  recovers the commutative description (1.9) while  $\Phi = 0$  is the NC description by (1.11). So we end up with the most general form of the equivalence for slowly varying fields, i.e.,  $\sqrt{\kappa}|\frac{\partial F}{F}|$ :

$$\hat{S}_\Phi(\mathcal{G}_s, \mathcal{G}, \hat{A}, \vartheta) = S(g_s, g, A, B) + \mathcal{O}(\sqrt{\kappa}\partial F), \quad (1.26)$$

which was proved by Seiberg and Witten [1] using the change of variables, (1.23) and (1.24), and the differential equation (1.21). Using the change of variables (1.23) and (1.24), we also get the analogue of Eq. (1.18)

$$S(\mathcal{G}_s, \mathcal{G}, A, \vartheta; \Phi) = \frac{2\pi}{\mathcal{G}_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{-\det(\mathcal{G} + F\vartheta\mathcal{G} + \kappa(\Phi + F\vartheta\Phi + F))}. \quad (1.27)$$

Note that the commutative action (1.27) is exactly the same as the DBI action obtained from the worldsheet sigma model using  $\zeta$ -function regularization scheme [10].

## 2 Exact Seiberg-Witten Map and Induced Gravity

In this section we will discuss the meaning of the equivalence (1.26) from the field theory point of view. The equivalence between the action (1.9), expressed in the form (1.27), and the action (1.25) immediately leads to <sup>3</sup>

$$\begin{aligned} \int d^{p+1}x \sqrt{-\det(\mathcal{G} + \kappa(\widehat{\mathcal{F}} + \Phi))} \\ = \int d^{p+1}x \sqrt{\det(1 + F\vartheta)} \sqrt{-\det(\mathcal{G} + \kappa(\Phi + \mathbf{F}))} + \mathcal{O}(\sqrt{\kappa}\partial F), \end{aligned} \quad (2.1)$$

where

$$\mathbf{F}_{\mu\nu}(x) = \left( \frac{1}{1 + F\vartheta} F \right)_{\mu\nu}(x). \quad (2.2)$$

What is the meaning of the equivalence (2.1)? First note that the left hand side of Eq. (2.1) is the NC description preserving NC gauge symmetry and  $\vartheta$  appears only in the  $\star$  product of the field strength  $\widehat{\mathcal{F}}_{\mu\nu}$ . On the other hand, the right hand side of Eq. (2.1) is the commutative description preserving ordinary gauge symmetry and  $\vartheta$  explicitly appears in the commutative DBI action. Next we see from the argument in section 4.1 in [1] that Eq. (2.1) is consistent with the differential equation (1.21) defining the map, e.g., Eqs. (1.14) and (1.15), between ordinary and NC gauge fields. Therefore we regard the right hand side of Eq. (2.1) as the SW map of the left hand side, i.e., NC DBI action labelled by the two-form  $\Phi$ , valid for every value of the parameters. We will illustrate this assertion for the case  $\Phi = 0$  and for  $p = 3$ , for definiteness, although our following argument also goes through for general cases. Note that Eq. (2.1) in the  $\Phi = B$  case where  $\vartheta = 0$  is a trivial identity since both sides are equally commutative descriptions.

For the case  $\Phi = 0$  and  $p = 3$ , the identity (2.1) reduces to

$$\int d^4x \sqrt{-\det(G + \kappa\widehat{F})} = \int d^4X \sqrt{\det(1 + F\theta)} \sqrt{-\det(G + \kappa\mathbf{F})}, \quad (2.3)$$

where we intentionally distinguished the commutative coordinates  $X$  from the NC ones for the following discussion. One can expand both sides of Eq. (2.3) in powers of  $\kappa$ .  $\mathcal{O}(1)$  implies that there is a measure change between NC and commutative descriptions

$$d^4x = d^4X \sqrt{\det(1 + F\theta)}. \quad (2.4)$$

In other words, the coordinate transformations,  $x^\mu \rightarrow X^\mu(x)$ , between NC and commutative descriptions depend on the dynamical gauge fields. Since the identity (2.3) must be true for arbitrary small  $\kappa$ , substituting Eq. (2.4) into Eq. (2.3) leads to the following relation

$$\widehat{F}_{\mu\nu}(x) = \left( \frac{1}{1 + F\theta} F \right)_{\mu\nu}(X). \quad (2.5)$$

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<sup>3</sup>The equivalence (2.1) was also proved in [15] in the framework of deformation quantization. We thank P. Schupp for drawing our attention to their paper.

Now  $\mathcal{O}(\kappa^2)$  in Eq. (2.3) leads to a remarkable identity

$$\begin{aligned} & -\frac{1}{4g_{YM}^2} \int d^4x \sqrt{-\det G} G^{\mu\alpha} G^{\nu\beta} \hat{F}_{\mu\nu} \star \hat{F}_{\alpha\beta} \\ &= -\frac{1}{4g_{YM}^2} \int d^4x \sqrt{-\det G} \sqrt{\det(1 + F\theta)} G^{\mu\alpha} G^{\nu\beta} \mathbf{F}_{\mu\nu} \mathbf{F}_{\alpha\beta} + \mathcal{O}(\sqrt{\kappa} \partial F), \end{aligned} \quad (2.6)$$

where we used the same symbol  $x$  again for both descriptions since the distinction is no longer necessary. Here we further discuss the ambiguity for constant  $\hat{F}$  mentioned in footnote 2. For the constant  $\hat{F}$  whose exact map is given by Eq. (1.22), the identity (2.6) is not quite true since the right hand side contains the additional factor  $\sqrt{\det(1 + F\theta)}$ . We see from Eq. (2.4) that the Jacobian factor for the coordinate transformation,  $x \rightarrow X(x)$ , precisely reproduces the additional factor.

We argued that Eq. (2.6) defines the exact nonlinear action of SW deformed electrodynamics. Note that the identity (2.6) holds for an arbitrary constant open string metric  $G_{\mu\nu}$ . For simplicity, we may take  $G_{\mu\nu} = \eta_{\mu\nu}$ , i.e., flat Minkowski spacetime. Eq. (2.6) then takes an interesting form

$$-\frac{1}{4g_{YM}^2} \int d^4x \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} = \frac{1}{4g_{YM}^2} \int d^4x \sqrt{\det(1 + F\theta)} \left( \frac{1}{1 + F\theta} F \frac{1}{1 + F\theta} F \right). \quad (2.7)$$

If we introduce an “effective non-symmetric metric” induced by dynamical gauge fields such that

$$g_{\mu\nu} = \eta_{\mu\nu} + (F\theta)_{\mu\nu}, \quad (g^{-1})^{\mu\nu} \equiv g^{\mu\nu} = \left( \frac{1}{1 + F\theta} \right)^{\mu\nu}, \quad (2.8)$$

the NC Maxwell action after the SW map formally looks like ordinary Maxwell theory coupled to the effective metric  $g_{\mu\nu}$ :

$$S = -\frac{1}{4g_{YM}^2} \int d^4x \sqrt{-\det g} g^{\mu\alpha} g^{\beta\nu} F_{\mu\nu} F_{\alpha\beta}. \quad (2.9)$$

It is easy to derive the exact equation of motion from the action (2.7) or (2.9)

$$\begin{aligned} \partial_\mu \left[ \sqrt{-g} \left\{ (\theta g^{-1})^{\mu\alpha} \text{Tr} (g^{-1} F g^{-1} F) - 2 \left( (\theta g^{-1} F g^{-1} F g^{-1})^{\mu\alpha} - (\theta g^{-1} F g^{-1} F g^{-1})^{\alpha\mu} \right) \right. \right. \\ \left. \left. + 2 \left( (g^{-1} F g^{-1})^{\mu\alpha} - (g^{-1} F g^{-1})^{\alpha\mu} \right) \right\} \right] = 0. \end{aligned} \quad (2.10)$$

Recently Rivelles observed [16] that the action for NC field theories after SW map can be regarded as an ordinary field theory coupling to a field dependent gravitational background. Our result genuinely realizes his intriguing idea. The linearized gravitational coupling of the action (2.9) exactly reproduces the result, Eq. (17), in [16]. It should be remarked that the gravitational field in the action (2.9) cannot be interpreted just as a fixed background since it depends on the dynamical gauge fields.

Now we will show that our result in Eq. (2.7) is consistent with the results in [17] where it was proved that the terms of order  $n$  in  $\theta$  in the NC Maxwell action via SW map form a homogeneous polynomial of degree  $n + 2$  in  $F$  (*Proposition 3.1*) and explicitly presented the deformed action up to order  $\theta^2$ . It is obvious that Eq. (2.7) satisfies their *Proposition 3.1*. The explicit form of the  $\theta$ -expanded action in [17] has the following expression using the matrix notation (1.19)<sup>4</sup>

$$S = \frac{1}{4g_{YM}^2} \int d^4x \text{Tr} \left( \left(1 + \frac{1}{2} \text{Tr} F\theta\right) F^2 - 2F\theta F^2 + F\theta F^2 \theta F + 2F\theta F\theta F^2 - \text{Tr}(F\theta) F\theta F^2 + \frac{1}{8} (\text{Tr} F\theta)^2 F^2 - \frac{1}{4} \text{Tr}(F\theta)^2 F^2 + \mathcal{O}(\theta^3) \right). \quad (2.11)$$

It is straightforward to reproduce the result (2.11) from Eq. (2.7) using the formulas

$$\begin{aligned} \sqrt{\det(1 + F\theta)} &= 1 + \frac{1}{2} \text{Tr} F\theta - \frac{1}{4} \text{Tr}(F\theta)^2 + \frac{1}{8} (\text{Tr} F\theta)^2 + \mathcal{O}(\theta^3), \\ \frac{1}{1 + F\theta} &= 1 - F\theta + (F\theta)^2 + \mathcal{O}(\theta^3). \end{aligned}$$

Another interesting case arises from the choice  $\Phi_{\mu\nu} = -B_{\mu\nu}$ , which naturally appears in Matrix models [1, 18]. In this case, using the metric  $g_{\mu\nu}$  with Euclidean signature instead,

$$\theta = \frac{1}{B}, \quad G = -\kappa^2 B \frac{1}{g} B, \quad G_s = g_s \sqrt{\det(\kappa B g^{-1})} \quad (2.12)$$

and

$$(\hat{F} + \Phi)_{\mu\nu} = iB_{\mu\lambda} [X^\lambda, X^\sigma]_\star B_{\sigma\nu}, \quad (2.13)$$

where

$$X^\mu = x^\mu + \theta^{\mu\nu} \hat{A}_\nu. \quad (2.14)$$

The DBI action related to Matrix models has a more natural description, the so-called background independent formulation, in terms of closed string variables [18]. Using the relations (2.12) and (2.13), the equivalence (1.26) can be recast as

$$\begin{aligned} & \frac{2\pi}{g_s(2\pi)^{\frac{p+1}{2}}} \int \frac{d^{p+1}x}{|\text{Pf}\theta|} \sqrt{\det(\delta_\mu^\nu - \frac{i}{\kappa} g_{\mu\lambda} [X^\lambda, X^\nu]_\star)} \\ &= \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{\det(1 - \hat{F}\theta)} \sqrt{\det(g + \kappa(B + \hat{\mathbf{F}}))}, \\ &= \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}} \int d^{p+1}x \sqrt{\det(g + \kappa(B + F))}, \end{aligned} \quad (2.15)$$

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<sup>4</sup>To avoid a confusion, we point out that  $F^2 = F_{\mu\nu} F^{\mu\nu}$  in [17] corresponds to our  $-F^2 = F_{\mu\nu} F^{\nu\mu}$  in the matrix notation (1.19).



where

$$\hat{\mathbf{F}}_{\mu\nu}(x) = \left( \frac{1}{1 - \hat{F}\theta} \hat{F} \right)_{\mu\nu}(x). \quad (2.16)$$

We point out that the identity in Eq. (2.15) is come out from the equivalence (1.26) by expressing the NC DBI action (1.25) in terms of closed string variables. One can check that our result (2.15) for slowly varying fields is consistent with the exact SW map obtained by completely independent way in [11, 12, 13, 14].

As was discussed in [18],  $X^\mu$  are background independent, i.e.,  $\theta$ -independent, coordinates and can be used to describe the coordinates on D-branes for all values of  $\theta$ . Thus one can see that both sides of Eq. (2.15) are background independent since the integral measure  $\frac{1}{(2\pi)^{\frac{p+1}{2}}} \int \frac{d^{p+1}x}{|\text{Pf}\theta|}$ , which is a trace over the Hilbert space of the algebra (1.8) [6], and  $F + B$  are background independent objects. Similarly to Eq. (2.1), Eq. (2.15) also defines a map between NC (Matrix) and commutative descriptions, but now in terms of inverse SW map.

Note that

$$B + \hat{\mathbf{F}} = \frac{1}{1 - \hat{F}\theta} B. \quad (2.17)$$

In the zero slope limit,  $\kappa \rightarrow 0$ , now keeping  $g_{\mu\nu}$  and  $g_{YM}^2$  fixed, Eq. (2.15) gives rise to an intriguing identity

$$\frac{1}{4g_{YM}^2} \int d^{p+1}x \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} = \frac{1}{4g_{YM}^2} \int d^{p+1}x \sqrt{\det \hat{\mathbf{g}}} \hat{\mathbf{g}}^{\mu\alpha} \hat{\mathbf{g}}^{\beta\nu} B_{\mu\nu} B_{\alpha\beta}, \quad (2.18)$$

where  $\mathcal{F} = B + F$  and

$$\hat{\mathbf{g}}_{\mu\nu} = \delta_{\mu\nu} - (\hat{F}\theta)_{\mu\nu}, \quad \hat{\mathbf{g}}^{\mu\nu} = \left( \frac{1}{1 - \hat{F}\theta} \right)^{\mu\nu}. \quad (2.19)$$

A naive interpretation of the identity (2.18) may be that fluctuations  $F$  with respect to the background  $B$  induce fluctuations of (NC) geometry from the Matrix model side. It will be very interesting if one can understand this picture from  $D$ -brane perspective.

### 3 Discussion

Our results in the present paper may have many interesting implications in both string theory and field theory. In particular we showed that the dual description via SW map describes a fluctuating geometry induced by gauge fields and noncommutativity, in a sense, reflects the presence of a fluctuating “medium”. The spacetime geometry (nonsymmetric gravity [19, 20, 21]) determined by the metric (2.8) does deserve further study. The geometry to the leading order in  $\theta$  was studied in [16] where it was shown that the plane wave solution in [22, 23, 24] corresponds to a geodesic motion of massless particles in that gravitational field.

There are several interesting open issues for the future: Non-Abelian generalization [25, 26, 27], an exact SW map for DBI actions with derivative corrections [28, 29, 30], and to find

an exact SW map for currents by incorporating matter fields [13, 14, 31]. It will surely be interesting to reexamine noncommutative  $U(1)$  instantons [32, 33, 34, 35, 36] in view of the action (2.9), which may be helpful to ponder topological issues in the commutative description via SW map. We will report our progress on these issues elsewhere in the near future.

**Notes added** Several points raised in this paper have been clarified since the original version of this paper was posted to the archive.

Our method to obtain the exact SW map is extremely simple. In particular, we got the SW maps for the measure change and the field strength, Eq. (2.4) and Eq. (2.5), respectively. It was shown in [37] that these maps can be derived from the equivalence between the star products  $\star_\omega$  and  $\star_B$  defined by the symplectic forms  $\omega = B + F$  and  $B$ , respectively, in the context of deformation quantization and one can find the exact SW map for an adjoint scalar field using the results. It was also shown there that topological invariants in NC gauge theory are mapped to the usual Chern classes via the exact SW map.

The closed form for the exact SW map of NC electrodynamics, Eq. (2.9), turned out to pose an important physics about emergent gravity [38, 39, 40]. It was shown in [38, 39] that NC  $U(1)$  instantons are equivalent to gravitational instantons via the exact SW map, indeed posed at the last of the Discussion above and speculated at the last paragraph in section 6 of [37]. In particular, we showed in [40] that self-dual electromagnetism in noncommutative spacetime is equivalent to self-dual Einstein gravity.

Using the exact SW maps presented in this paper and [37], Mukherjee and Saha showed [41, 42] that either NC Chern-Simons or NC Maxwell-Chern-Simons model with scalar matter in the adjoint representation and without any potential term does not have any nontrivial BPS soliton in the sector which has a smooth commutative limit. Their results clearly show that, in these models, there is no non-trivial, non-perturbative solution depending on the NC parameter and vanishing smoothly along with it, which is consistent with the topological property of soliton solutions.

It was also discussed in [43] that NC field theory interpreted as ordinary field theory embedded in a gravitational background induced by gauge fields is quite similar to quantum field theory coupling to a non-symmetric metric background at the phenomenological level and the two theories predict experimentally measurable consequences in the high energy process such as the pair annihilation  $e^+e^- \rightarrow \gamma\gamma$ . The geometry generated by NC gauge fields was further studied in [44] for the massive Klein-Gordon field.

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